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Bounds for zeros of the Laguerre polynomials

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Abstract

We establish new inequalities on the extreme zeros of the Laguerre polynomials which are uniform in all the parameters involved.

1. Introduction

In this note, we establish explicit bounds, uniform in k and α, on the extreme zeros of the Laguerre polynomials $L_k^{(\alpha)}(x)$. For our purposes, it will be convenient to define them as a nonzero polynomial solution of

$$xy'' = (x - \alpha - 1)y' - ky. \tag{1}$$

A well-known upper bound for the largest zero of $L_k^{(\alpha)}(x)$, provided $|\alpha| \geqslant \frac{1}{4}, \ \alpha > -1$, is

$$x_k < (\sqrt{4k + 2\alpha + 2} - 6^{-1/3}(4k + 2\alpha + 2)^{-1/6} i_{11})^2, \tag{2}$$

where i_1 is the smallest zero of the Airy function [11], and, as Szegö pointed out, the constant $6^{-1/3} i_{11} = 1.85575\ldots$, cannot be replaced by a smaller one. As a matter of fact (not as well known as it should be) the last claim is true only if α is fixed. The best currently known inequality for the least zero belongs to Ismail and Li [6]. They proved that for $\alpha > -1$, all the zeros of $L_k^{(\alpha)}(x)$ are in the interval

$$2k + \alpha - 2 \pm \sqrt{1 + 4(k-1)(k+\alpha-1)\cos^2 \frac{\pi}{k+1}}. \tag{3}$$

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Surprisingly enough, it seems that the general asymptotic of the extreme zeros (uniform in α) is still unknown. Some related results for α varying with k can be found in [1,2,5]. Here we use a technique suggested in [4,8] to prove the following sharper inequalities.

Theorem 1. *Let x_1 and x_k be the least and the largest zeros of $L_k^{(\alpha)}(x)$, respectively. For $k \geqslant 7,\ \alpha \geqslant 8$, the following inequalities hold*:

$$x_1 > s - r + \frac{(s-r)^{2/3}}{2r^{1/3}}, \tag{4}$$

$$x_k < s + r + \frac{(s+r)^{2/3}}{2r^{1/3}}, \tag{5}$$

where

$$s = 2k + \alpha + 1, \quad r = \sqrt{4k^2 + (2k-1)(2\alpha+2)}.$$

More precisely, all the zeros of $L_k^{(\alpha)}(x)$ are confined between the only two real roots of the following equation:

$$(x^2 - 2sx + b^2 - 1)^3 - 4sx^3 + 9s^2x^2 + (b^2-1)(b^2 - 1 - 6sx) = 0, \tag{6}$$

where $b = \alpha + 2$.

It looks plausible that, up to the factor $\frac{1}{2}$, Theorem 1 gives the correct value of the second term of the corresponding asymptotics.

2. Proofs

A real entire function $\phi(x)$ is in the Laguerre–Polya class $\mathscr{L}$–$\mathscr{P}$ if it has a representation of the form

$$\phi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) e^{-x/x_k} \quad (\omega \leqslant \infty),$$

where c, β, x_k are real, $\alpha \geqslant 0$, m is a nonnegative integer and $\sum x_k^{-2} < \infty$. Our main tool will be the following inequality valid for any $f \in \mathscr{L}$–$\mathscr{P}$ [7,9,10],

$$V_m(f(x)) = \sum_{j=-m}^{m} (-1)^{m+j} \frac{f^{(m-j)}(x) f^{(m+j)}(x)}{(m-j)!(m+j)!} \geqslant 0, \quad m = 0, 1, \ldots \ . \tag{7}$$

We will use $m = 2$ and set

$$V = 12V_2(y) = 3y''^2 - 4y'y''' + yy^{(4)}.$$

Notice that in our case, the positivity (and a plausible connection with the potential theory) can be seen directly by $V = \sum_{i \neq j} (x - x_i)^{-2} (x - x_j)^{-2}$, where $x_1, x_2, \ldots$ are the zeros of y [3].

In the sequel, we deal with the function $t = t(x) = y'/y$, and set $b,\ r$, and s as in Theorem 1 to simplify some expressions. We also assume $x>0$. Using differential equation (1) recursively to express the higher derivatives in V through y and y' we get

$$\frac{2x^3}{y^2} V = At^2 + 2Bt + C, \tag{8}$$

where

$$A = -2x(x^2 - 2sx + (b-1)(b+3)),$$
$$B = x^3 - (2s + b - 1)x^2 + (2bs - 3s + (b-1)(b+3))x + b - b^3,$$
$$C = (b - s - 1)(x^2 - 2sx - x + b^2 + b).$$

Observe that A is positive only for x in the interval (x^a_m, x^a_M),

$$x^a_{m,M} = 2k + \alpha + 1 \pm 2\sqrt{k^2 + \alpha k + k - \alpha - 1}.$$

Let also $x^c_m < x^c_M$ be the roots of C.

For the discriminant of the equation $At + 2Bt + C = 0$, in t we get

$$\begin{aligned}\Delta(x) &= B^2 - AC\\ &= (x^2 - 2sx + b^2 - 1)^3 - 4sx^3 + 9s^2x^2 + (b^2 - 1)(b^2 - 1 - 6sx)\end{aligned} \tag{9}$$

that is exactly expression (6).

We split the proof into several lemmas. We used Mathematica for symbolic calculations.

Lemma 1. *The equation $\Delta(x) = 0$ has exactly two real roots $x^*_m < x^*_M$, provided $k \geqslant 2$ and $\alpha > -1$. Moreover, $x^c_m < x^a_m < x^*_m < x^*_M < x^a_M < x^c_M$, if $k \geqslant 7,\ \alpha \geqslant 8$.*

Proof. The discriminant surface of (9) (i.e. the domain of parameters where the equation has multiple zeros) is given by the equation

$$(2k^2 + 2ak + 2k - a - 1)(a+2)^2 \times\ (k(k-1)(k+a+1)(k+a+2)(a+1)(a+3))^3 = 0.$$

Thus, for $k>1$ and $a > -1$, the number of real roots does not depend on a and k. Therefore, it is enough to check the claim for $k = 2,\ a = 0$, i.e. for the equation

$$x^6 - 30x^5 + 309x^4 - 1200x^3 + 1152x^2 - 360x + 36 = 0,$$

what is straightforward. Since $C/k - A/x = x + a - 1$, the zeros of $A/x = 0$ are confined between the zeros of $C = 0$, for $\alpha \geqslant 1$. To prove $x^a_m < x^*_m < x^*_M < x^a_M$, we consider the resultant of Δ and $A/2x$ in x,

$$Result(\Delta, A/2x) = (1+\alpha)^2(1 - \alpha - \alpha^2 + \alpha^3 + 7s^2 - \alpha s^2)^2.$$

Since it does not vanish for $k \geqslant 7,\ \alpha \geqslant 8$, it will be enough to check the claim for $k = 7,\ \alpha = 8$. As in this case both $\Delta(x^a_m), \Delta(x^a_M) > 0$, we are done. $\square$

Lemma 2. *For $k \geqslant 7$, $\alpha \geqslant 8$, all the zeros of $L_k^{(\alpha)}(x)$ are confined in (x_m^*, x_M^*), between the only two real roots of the equation $\Delta(x) = 0$.*

Proof. We prove $x_m^* < x_1$, the inequality $x_k < x_M^*$ is similar. Let $t_1 = t_1(x)$ be the solution of $At^2 + 2Bt + C = 0$, given by $t_1 = \frac{-B+\sqrt{B^2-AC}}{A}$. Then for $\alpha \geqslant 1$, t_1 is a continuous function on $(0, x_m^*]$ and $\lim_{x \to 0^+} t_1 = -\infty$. The first claim follows from $B^2 - (B^2 - AC) = AC > 0$, on rewriting t_1 as $\frac{-C}{B+\sqrt{B^2-AC}}$. The second one is trivial. On the other hand, by $V_1(y) = -y^2 t'(x) \geqslant 0$, $t(x)$ is a continuous decreasing function on $[0, x_1)$, tending to $-\infty$ for $x \to x_1^{(-)}$. It does not intersect the solutions of $At^2 + 2Bt + C = 0$, in particularly t_1. By the previous lemma this is possible only if $x_m^* < x_1$. □

Proof of Theorem 1. By the previous lemma it is enough to show that inequalities (4) and (5) hold for x_m and x_M, respectively. Since

$$s - r + \frac{(s-r)^{2/3}}{2r^{1/3}} < s < s + r + \frac{(s+r)^{2/3}}{2r^{1/3}},$$

and

$$\Delta(s) = -(s^2 - b^2 + 1)((s^2 - b^2)^2 - 3s^2 - b^2) < 0,$$

to prove (4) we just check

$$\Delta\left(s \pm r + \frac{(s \pm r)^{2/3}}{2r^{1/3}}\right) > 0.$$

Calculations yield

$$\begin{aligned} &\frac{64r^6}{q^4}\Delta\left(s - r + \frac{(s-r)^{2/3}}{2r^{1/3}}\right) \\ &\quad = q^8 - 32q^5 r - 60q^6 r^{4/3} - 32q^2 r^2 + 96q^3 r^{4/3} \\ &\qquad + 240q^4 r^{8/3} + 144r^{10/3} + 192qr^{11/3}, \end{aligned}$$

where $q = (s-r)^{1/3}$. Since, as it is easy to check, $r^2 > s + r > q^3$, and so

$$q^5 r < qr^{11/3}, \quad q^6 r^{4/3} < q^4 r^{8/3}, \quad q^2 r^2 < r^{10/3},$$

we convince that the above expression is positive. Hence (4) follows.

The proof of (5) is similar using $r^2 > s + r$, we omit the details. □

3. Final remarks

It is worth noticing that one can use as well the corresponding three-term recurrence instead of differential equation (1). Patrick [10] used (7) to obtain the

Turan-type inequalities, which have essentially the same form

$$U_m(f(x)) = \sum_{j=-m}^{m} (-1)^j \frac{p_{k-j}(x)p_{k+j}(x)}{(m-j)!(m+j)!} \geqslant 0, \quad m = 0, 1, \dots,$$

and hold for polynomials and entire functions having generating function $f(x) = \sum_{i=0}^{\infty} p_i \frac{z^i}{i!}$, of the Laguerre–Polya class. This is the case for the Laguerre polynomials as they have a generating function of the form

$$\sum_{i=0}^{\infty} \frac{L_i^{(\alpha)}(x)}{L_i^{(\alpha)}(0)} \frac{z^i}{i!} = \frac{J_\alpha(2\sqrt{xz})}{(xz)^{\alpha/2}} \Gamma(\alpha+1)e^z, \quad \alpha > -1.$$

With $m = 2$, it gives practically the same bounds as Theorem 1.

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